USING MATHEMATICA TO OBTAIN EXPECTED VALUES OF SUMS OF SQUARES

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1. INTRODUCTION

Expected values of mean squares from the following experiment designs are considered here:

i) split plot experiment design, Table 1;

ii) resolvable incomplete block design for $v = k^2$ treatments in incomplete blocks of size k with r replicates, Table 2;

iii) resolvable incomplete block experiment design for v = bk treatments in b incomplete blocks of size k in each of r replicates, Table 3;

iv) incomplete block or lattice square experiment design with $v = k^2$ treatments in incomplete blocks of size k in r replicates with differential trends or gradients in each incomplete block, Table 4;

and

v) lattice square experiment design with $v = k^2$ treatments in each replicate, Table 5.

2. SPLIT-PLOT EXPERIMENT DESIGN

To illustrate the program, Table 1, for obtaining expected values of mean squares in an analysis of variance, ANOVA, we use a classical split-plot design where the w whole plot treatments are in a randomized complete block experiment design, RCBED, and the s split plots are randomly allotted to the split-plot experimental units within *each* whole plot. The ANOVA for whole plots is the standard RCBED one for the response model

$$Y_{ghi} = \mu + \rho_g + \tau_h + \delta_{gh} + \alpha_i + \alpha\tau_{hi} + \epsilon_{ghi} , \tag{2.1}$$

where μ is a general mean effect, ρ_g is the effect of the g^{th} replicate, $g = 1,\cdots,r$, τ_h is the effect of the h^{th} whole plot treatment, $h = 1,\cdots,w$, δ_{gh} is a random error effect distributed with mean zero and variance σ_δ^2, α_i is the effect of the i^{th} split-plot treatment, $i = 1,\cdots,s$, $\alpha\tau_{hi}$ is the interaction effect of the h^{th} whole plot and i^{th} split-plot treatments, and ϵ_{ghi} is a random error effect distributed with mean zero and variance σ_ϵ^2.

For the program in Table 1, the following notations and functions from MATHEMATICA were used:

r=2 -- assigning value 2 to r (number of replicates), the single "=" sign denotes assignment.

; -- the semicolon at the end of each command suppresses output printing of that command.

Array[Y, {r, w, s}] -- creating an array Y with dimension $r \times w \times s$, where r, w and s should have been assigned with specific values.

Y[g, h, i] -- Y_{ghi}, square brackets and commas are necessary here to specify indices of array Y, where g, h and i are fixed values.

Y[g_, h_, i_] -- also indicating Y_{ghi}, but the underscore following index g, h, or i allows the index to take on any number assigned to them, i.e., g_, h_ and i_ are variables.

d[g_, h_] d[g_, h_] -> D -- the arrow sign "->" denotes constraint, so this command constrains the product d_{gh}^2 to the value D; note that the space between the two d[g_, h_]'s indicates a multiplication.

Sum[Y[g, h, i], {i, 1, s}] -- summing Y_{ghi} over index i with the range from 1 to s, i.e., $\sum_{i=1}^{s} Y_{ghi}$.

^2 -- the notation "^" denotes exponent, so ^2 indicates a power of 2.

Expand[A] /. res -- the notation "/." denotes restrictions, so this command expanding the expression of A under the restriction res.

If a command is longer than one line, one can split it into several lines, but the end of each line, except the last line, must end with an operator such as +, −, /, period (which also signifies multiplication) or a comma.

In this example the restriction, res, is used for the independence of effects and expected value of squares of effects. The expected value of the following error (A) sum of squares is obtained by the MATHEMATICA program given in Table 1:

$$\sum_{g=1}^{r}\sum_{h=1}^{w}\left(\sum_{i=1}^{s} Y_{ghi}\right)^2 \Big/ s - \sum_{g=1}^{r}\left(\sum_{h=1}^{w}\sum_{i=1}^{s} Y_{ghi}\right)^2 \Big/ ws - \sum_{h=1}^{w}\left(\sum_{g=1}^{r}\sum_{i=1}^{s} Y_{ghi}\right)^2 \Big/ rs + \left(\sum_{g=1}^{r}\sum_{h=1}^{w}\sum_{i=1}^{s} Y_{ghi}\right)^2 \Big/ rws$$

$$= \sum_{g}\sum_{h} Y_{gh\cdot}^2 \Big/ s - \sum_{g} Y_{g\cdot\cdot}^2 \Big/ ws - \sum_{h} \frac{Y_{\cdot h\cdot}^2}{rs} + \frac{Y_{\cdot\cdot\cdot}^2}{rws}. \tag{2.2}$$

Under the specified restriction, the expected value of the error A sum of squares is

$$(r-1)(w-1)\Big(E + AD\Big) = 2\left[\sigma_\epsilon^2 + 4\sigma_\delta^2\right]. \tag{2.3}$$

3. INCOMPLETE BLOCK EXPERIMENT DESIGN

It is desired to obtain the expected value of blocks (eliminating treatment effects) mean square under the linear model

$$Y'_{ghi} = \mu + \rho_g + \beta_{gh} + \tau_i + \epsilon_{ghi} , \tag{3.1}$$

where μ is a general mean effect, ρ_g is the effect of replicate $g = 1,\cdots,r$, β_{gh} is the effect of block h in replicate g, $h = 1,\cdots,k$, τ_i is the effect of treatment $i = 1,\cdots,v$, β_{gh} is IID with zero mean and variance σ^2_β, and ϵ_{ghi} is IID with zero mean and variance σ^2_ϵ.

In solving for the effects, we use the constraints:

$$\sum_{g=1}^{r} \hat{\rho}_g = \sum_{h=1}^{b} \hat{\beta}_{gh} = \sum_{i=1}^{v} \hat{\tau}_i = 0 . \tag{3.2}$$

Also, since the β_{gh} and τ_i are orthogonal to replicates and the overall mean, we use $Y'_{ghi} - \bar{y}_{g\cdot\cdot} = Y_{ghi}$ as our observations in order to have a simpler form of the normal equations. Note that this procedure is not usable when treatment and block effects are not orthogonal (e.g., missing observations) to mean and replicate effects.

The normal equations for blocks and treatments are:

$$\begin{bmatrix} k\mathbf{I}_{rb} & \mathbf{N}_{rb\times v} \\ \mathbf{N}'_{v\times rb} & r\mathbf{I}_v \end{bmatrix} \begin{bmatrix} \beta_{rb\times 1} \\ \tau_{v\times 1} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{rb\times 1} \\ \mathbf{T}_{v\times 1} \end{bmatrix} , \tag{3.3}$$

where $\mathbf{I}_X$ is the identity matrix of order x, $\mathbf{N} = \{n_{ghi}\}$ is the block by treatment design matrix with rb rows and v columns, and $\mathbf{N}$ = NP in Table 2 and NT in Table 3. $\beta_{rb\times 1}$ is a column vector of ρ_{gh} (block effects), τ is a column vector of τ_i (treatment effects), B is a column vector of B_{gh} (block totals for the $Y_{ghi} = Y'_{ghi} - \bar{y}_{g\cdot\cdot}$ values), and $\mathbf{T}$ is a column vector of treatment totals for the Y_{ghi} values. The matrix equation for blocks eliminating treatment effects is

$$\left(k\mathbf{I}_{rb} - \mathbf{N}\ \mathbf{N}'/r\right)\beta = \mathbf{B} - \mathbf{N}\ \mathbf{T}/r . \tag{3.4}$$

Using the constraint in (3.2) for blocks, solutions for β may be found. There are many ways of effecting a solution for (3.4) using (3.3). One method is to add $\mathbf{J0}/r$ to $(k\mathbf{I}_{rb} - \mathbf{N}\ \mathbf{N}'/r)$, where

$$\mathbf{J0} = \begin{bmatrix} \mathbf{0}_{k\times k} & \mathbf{J}_{k\times k} & \cdots & \mathbf{J}_{k\times k} \\ \mathbf{J}_{k\times k} & \mathbf{0}_{k\times k} & \cdots & \mathbf{J}_{k\times k} \\ \vdots & & & \\ \mathbf{J}_{k\times k} & \mathbf{J}_{k\times k} & \cdots & \mathbf{0}_{k\times k} \end{bmatrix} . \tag{3.5}$$

where $\mathbf{0}_{k\times k}$ is a matrix of zeros and $\mathbf{J}_{k\times k}$ is a matrix of ones. Then the solution is obtained as

$$\hat{\beta} = \left[k\mathbf{I}_{rk} - \mathbf{N}\ \ \mathbf{N}'/r + \mathbf{J0}/r\right]^{-1}\left[\mathbf{B} - \mathbf{N}\ \ \mathbf{T}/r\right]. \tag{3.6}$$

Note that

$$\mathbf{Var} = \left[k\mathbf{I}_{rk} - \mathbf{N}\ \ \mathbf{N}'/r + \mathbf{J0}/r\right]^{-1}\sigma_\epsilon^2 = \mathbf{M}\sigma_\epsilon^2\,, \tag{3.7}$$

is a form of the variance-covariance matrix for $\hat{\boldsymbol{\beta}}$. Using **J0** invokes the constraints $\sum_{k=1}^{k}\hat{\beta}_{gh} = 0$ for each g. The form of $\mathbf{N}\ \ \mathbf{N}'/r - \mathbf{J0}/r$ will be diagonal for $v = k^2$.

The MATHEMATICA program in Table 2 may be used to evaluate the expected value of the block (eliminating treatment effects) sum of squares,

$$\hat{\beta}'\left[\mathbf{B} - \mathbf{N}\ \ \mathbf{T}/r\right]. \tag{3.8}$$

3.1. $v = k^2$ Treatments, r Replicates, rk Blocks

The sum of squares for blocks (eliminating treatment effects) is SSB:

$$\mathrm{SSB} = \hat{\beta}'\left[\mathbf{B} - \mathbf{N}\ \ \mathbf{T}/r\right] = \left[\mathbf{B} - \mathbf{N}\ \ \mathbf{T}/r\right]'\mathbf{M}\left[\mathbf{B} - \mathbf{N}\ \ \mathbf{T}/r\right]. \tag{3.9}$$

The mean square is $\mathrm{SSB}/r(k-1)$ where $r(k-1)$ is the number of degrees of freedom associated with SSB. If **Var** is diagonal, then one may simply obtain the expectation of the term

$$\left(Y_{gh\cdot} - \sum_{i=1}^{v} n_{ghi}\bar{y}_{\cdot\cdot i}\right)^2. \tag{3.10}$$

Multiply this value by the trace of Var, and divide by degrees of freedom to obtain the expected value of $\mathrm{SSB}/r(k-1)$ which is

$$\sigma_\epsilon^2 + (r-1)k\sigma_\beta^2/r\,. \tag{3.11}$$

When **M** is of the diagonal form, the expectation may be programmed easily in GAUSS.

The coefficient of σ_β^2 in the expected value of (3.9) is $k(k-1)(1-1/r)^2$ for each gh. There are rk such terms, the diagonal element of **M** is $r/k(r-1)$, and there are $r(k-1)$ degrees of freedom. Therefore, the coefficient for σ_β^2 in the expected value of $\mathrm{SSB}/r(k-1)$ is

$$\frac{rk}{r(k-1)}\left(\frac{r}{k(r-1)}\right)k(k-1)(1-1/r)^2 = (r-1)k/r\,. \tag{3.12}$$

The result in (3.11) and (3.12) agrees with that given in Kempthorne (1952) and Federer (1955). For the simple or double lattice this is $k/2$, for the triple lattice it is $2k/3$, and for the balanced lattice, it is $k^2/k+1)$.

MATHEMATICA may be used to obtain the expected value of (3.9) for specific values of v, k, and r. To illustrate the procedures, we used $v = 4$, $k = 2$, and $r = 3$ for the program in Table 2. Some new notations and functions from MATHEMATICA were used:

B={b11, b12, b21, b22, b31, b32} -- assigning a vector with specified elements b11,...,b32 to B.

NB={{1,1,0,0}, {0,0,1,1}, {1,0,1,0}, {0,1,0,1}, {1,0,0,1}, {0,1,1,0}} -- assigning a matrix with specified elements to NB, where each set of inner braces encloses a row of the matrix.

IdentityMatrix[v] -- creating an identity matrix of order v.

Table[1, {i,1,v}, {j,1,v}] -- creating a two-way table (matrix) with v rows and v columns, and all elements are set to 1.

Inverse[X] -- obtaining the inverse of matrix X.

Transpose[X] -- obtaining the transposed matrix of X.

Note that the matrix **Jrk** could also be created as a Kronecker product of the two matrices **Ir** (identity matrix of order r) and **Jk** ($k \times k$ matrix of ones) with the function Outer[Times, **Ir**, **Jk**]. In this example res is the set of rules to follow, e.g., all cross-products of effects are zero and all squares have positive values, say E and R here. The expected value of (3.9) is $r(k-1)\left(\sigma_\epsilon^2 + k(r-1)\sigma_\beta^2/r\right) = 3(E + 4R/3)$ as given in Table 2.

3.2. $v = kb$, r Replicates, b Incomplete Blocks of Size k, $k \leq b$

When $k \leq b$, the procedure using the trace described in the previous section does not work since adding **J0** does not produce a diagonal form of the variance-covariance matrix, i.e.,

$$\mathbf{N}\ \mathbf{N}'/r - \mathbf{J0}/r$$

produces a matrix of the form

$$\begin{pmatrix} k\mathbf{I}_b & \mathbf{M}_{b\times b} & \mathbf{L}_{b\times b} & \mathbf{K}_{b\times b} & \cdots \\ \mathbf{M}'_{b\times b} & k\mathbf{I}_b & \mathbf{P}_{b\times b} & \mathbf{R}_{b\times b} & \cdots \\ \mathbf{L}'_{b\times b} & \mathbf{P}'_{b\times b} & k\mathbf{I}_b & \mathbf{R}_{b\times b} & \cdots \\ \vdots & & & & \end{pmatrix} \tag{3.13}$$

where the off-diagonal sub-matrices have a minus one in $(b-k)$ transversals of the $b \times b$ sub-matrix.

This means that the expected value of cross-products will be non-zero. This does not allow one to use the trace procedure of the previous section. Instead, it is necessary to obtain the expected value of

$$[\mathrm{B-N\ T/r}]'\mathbf{M}[\mathrm{B-N\ T/r}], \tag{3.14}$$

where **M** is a form of the variance-covariance matrix divided by σ_ϵ^2, depending upon how the restrictions **J0** were applied. However, the MATHEMATICA program for evaluating the expected value of the sum of squares in (3.14) is much the same as in Table 2 and is presented in Table 3 using v = 6, k = 2, b = 3, and r = 3. The main difference from Table 2 is how the program was written.

The expected value of (3.14) is

$$\mathrm{r(b-1)}\left(\sigma_\epsilon^2+\mathrm{k(r-1)}\sigma_\beta^2/\mathrm{r}\right) = 6(\mathrm{E}+4\mathrm{R}/3)\ , \tag{3.15}$$

i.e., the coefficient of $\hat{\sigma}_\beta^2 = \mathrm{R}$ in the mean squares is $8/6 = 4/3$ and the coefficient of $\hat{\sigma}_\epsilon^2 = \mathrm{E}$ is one as it must be.

4. INCOMPLETE BLOCK OR LATTICE SQUARE EXPERIMENT DESIGNS WITH DIFFERENTIAL TRENDS IN BLOCKS

If gradients or trends within incomplete blocks or within the rows (columns) of a lattice square experiment design and if the gradients vary randomly from block to block, an efficient method of statistical analysis is to recover the information present in much the same manner as recovery of inter-block information. A linear model for this situation is

$$\mathrm{Y}_{ghi} = \mu + \rho_g + \beta_{gh} + \gamma_{gh}\mathrm{A}_{ghi} + \tau_i + \epsilon_{ghi}\ , \tag{4.1}$$

where the A_{ghi} refer to the centered values for position within a block, γ_{gh} is a linear regression coefficient (note that a set of quadratic coefficients could be added also) for block gh and is distributed with zero mean and variance σ_γ^2, and the other values are defined as for (3.1). The normal equations for block, gradient, and treatment effects after eliminating mean and replicate effects are:

$$\begin{bmatrix} \mathrm{kI}_{rk} & 0_{rk\times rk} & \mathrm{NB}_{rk\times v} \\ 0 & \sum \mathrm{A}_{ghi}^2\mathrm{I}_{rk} & \mathrm{NG}_{rk\times v} \\ \mathrm{NB}' & \mathrm{NG}' & \mathrm{rI}_v \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ \tau \end{bmatrix} = \begin{bmatrix} \mathrm{B} \\ \mathrm{G} \\ \mathrm{T} \end{bmatrix}. \tag{4.2}$$

$\mathbf{0}_{rk\times rk}$ is a zero matrix since $\sum_i A_{ghi} = 0$ for every gh and $\sum_k A^2_{ghi} = \tau$, a constant, for every gh. Since gradients and blocks are orthogonal, the solutions for block, gradient, and treatment effects are:

$$\hat{\beta} = \left[k\mathbf{I}_{rk} - \mathbf{NB}(r\mathbf{I}_v - \mathbf{NG}'\ \mathbf{NG}/c)^{-1}\mathbf{NB}' + \mathbf{J0}/r\right]^{-1}$$

$$\times\left[\mathbf{B} - \mathbf{NB}(r\mathbf{I}_v - \mathbf{NG}'\ \mathbf{NG}/c)^{-1}(\mathbf{T} - \mathbf{NG}'\ \mathbf{G}/c)\right], \tag{4.3}$$

$$\hat{\gamma} = \left[c\mathbf{I}_{rk} - \mathbf{NG}(r\mathbf{I}_v - \mathbf{NB}'\ \mathbf{NB}/k + \mathbf{J}_v/k)^{-1}\mathbf{NG}'\right]^{-1}$$

$$\times\left[\mathbf{G} - \mathbf{NG}(r\mathbf{I}_v - \mathbf{NB}'\ \mathbf{NB}/k + \mathbf{J}_v/k)^{-1}(\mathbf{T} - \mathbf{NB}'\ \mathbf{B}/k)\right], \tag{4.4}$$

and

$$\hat{\tau} = \left[r\mathbf{I}_v - \mathbf{NG}'\ \mathbf{NG}/c - \mathbf{NB}'\ \mathbf{NB}/k + \mathbf{J}_v/k\right]^{-1}$$

$$\times\left[\mathbf{T} - \mathbf{NG}'\ \mathbf{G}/c - \mathbf{NB}'\ \mathbf{B}/k\right]. \tag{4.5}$$

The two sums of squares whose expected values are obtained from the program in Table 4 are block (eliminating all other effects) and gradients (eliminating all other effects). The block (eliminating treatment and gradient effects) sum of squares is

$$\hat{\beta} = \left[\mathbf{B} - \mathbf{NB}(r\mathbf{I}_v - \mathbf{NG}'\ \mathbf{NG}/c)^{-1}(\mathbf{T} - \mathbf{NG}'\ \mathbf{G}/c)\right]^{-1} \tag{4.6}$$

and gradient (eliminating treatment and block effects) is

$$\hat{\gamma} = \left[\mathbf{G} - \mathbf{NG}(r\mathbf{I}_v - \mathbf{NB}'\ \mathbf{NB}/k + \mathbf{J}_v/k)^{-1}(\mathbf{T} - \mathbf{NB}'\ \mathbf{B}/k\right]. \tag{4.7}$$

The nonorthogonality between blocks and gradients arises because they both are nonorthogonal to treatments even though orthogonal to each other in the absence of treatments. However, their joint orthogonality allows for the simpler matrix equations (4.3) to (4.7) rather than the more complicated ones used in Table 5.

The sum of squares in (4.6) has $r(k-1)$ degrees of freedom, and the one in (4.7) has rk degrees of freedom. For $v = 9$, $k = 3$, and $r = 3$, the expected values of these two sums of squares is obtained as shown in Table 4. The coefficient of $\hat{\sigma}^2_\epsilon = E$ is $r(k-1) = 6$ for (4.5) and $rk = 9$ for (4.6). The coefficient of $\hat{\sigma}^2_\beta$ appears to have a difficult analytic solution, while that for $\hat{\sigma}^2_\gamma = GR$ appears to be

$$\Big(k(r-1)-1\Big)c/rk = 10/9\ . \tag{4.8}$$

The programming to obtain the expected values for the various sums of squares is somewhat more complex than that in Tables 2 and 3. The format is somewhat different but the ideas are similar. Restrictions for (4.6), i.e., rbd, are different than those, rgs, for (4.7) since cross-products and squares will differ.

5. LATTICE SQUARE OR LATTICE RECTANGLE EXPERIMENT DESIGNS

A linear response model for the designs of this class is

$$Y_{ghij} = \mu + \rho_g + \beta_{gh} + \gamma_{gi} + \tau_j + \epsilon_{ghij}\,, \tag{5.1}$$

where μ is a general mean effect, ρ_g is the g^{th} replicate effect, β_{gh} is the h^{th} row random effect in the g^{th} replicate, γ is the i^{th} column random effect in the g^{th} replicate, τ_j is the effect of treatment j, and ϵ_{ghij} is a random error effect. To recover row and column information, expected values of row (eliminating treatment and column effects) and of column (eliminating treatment and row effects) are needed. Although these are available for lattice square experiment designs, those for resolvable lattice rectangle designs may be unknown.

The normal equations after eliminating mean and replicate effects are :

$$\begin{bmatrix} k\mathbf{I}_{rk} & \mathbf{RC}_{rk \times rk} & \mathbf{RT}_{rk \times v} \\ \mathbf{RC}' & k\mathbf{I}_{rk} & \mathbf{CT}_{rk \times v} \\ \mathbf{RT}' & \mathbf{CT}' & r\mathbf{I}_v \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ \tau \end{bmatrix} = \begin{bmatrix} \mathbf{R} \\ \mathbf{U} \\ \mathbf{T} \end{bmatrix}, \tag{5.2}$$

where **RC** is the row by column incidence matrix and has the Kronecker product of $\mathbf{I}_r$ and $\mathbf{J}_{k \times k}$ form, **RT** is the row by treatment incidence matrix, **CT** is the column by treatment incidence matrix, **R**, **U**, and **T** are the column vectors of totals for rows, columns, and treatments, respectively (U was used instead of **C** because MATHEMATICA had it "protected"). Using the usual restraints, solutions for the various effects are:

$$\hat{\beta} = \left[k\mathbf{I}_{rk} - (\mathbf{Z} \quad \mathbf{RT}) \begin{pmatrix} k\mathbf{I}_{rk} & (\mathbf{CT}-\mathbf{J})/k \\ \mathbf{CT}' & r\mathbf{I}_v \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{RC}' \\ \mathbf{RT}' \end{pmatrix} + \mathbf{J0}/r \right]^{-1}$$

$$\times\left[\mathbf{R}-(\mathbf{Z}\quad \mathbf{RT})\begin{pmatrix} \mathrm{k}\mathbf{I}_{rk} & (\mathbf{CT}-\mathbf{J})/\mathrm{k} \\ \mathbf{CT}' & \mathrm{r}\mathbf{I}_v \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{U} \\ \mathbf{T} \end{pmatrix}\right], \tag{5.3}$$

and

$$\hat{\gamma} = \left[\mathrm{k}\mathbf{I}_{rk}-(\mathbf{Z}\quad \mathbf{CT})\begin{pmatrix} \mathrm{k}\mathbf{I}_{rk} & (\mathbf{RT}-\mathbf{J})/\mathrm{k} \\ \mathbf{RT}' & \mathrm{r}\mathbf{I}_v \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{RC}' \\ \mathbf{CT}' \end{pmatrix} + \mathbf{J0}/\mathrm{r}\right]^{-1}$$

$$\times\left[\mathbf{U}-(\mathbf{Z}\quad \mathbf{CT})\begin{pmatrix} \mathrm{k}\mathbf{I}_{rk} & (\mathbf{RT}-\mathbf{J})/\mathrm{k} \\ \mathbf{RT}' & \mathrm{r}\mathbf{I}_v \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{R} \\ \mathbf{T} \end{pmatrix}\right]. \tag{5.4}$$

where Z is a matrix of zeros. The row (eliminating treatment and column effects) sum of squares is

$$\hat{\beta}'\left[\mathbf{R}-(\mathbf{Z}\quad \mathbf{RT})\begin{pmatrix} \mathrm{k}\mathbf{I}_{rk} & \mathbf{CT}-\mathbf{J}_{rk\times v} \\ \mathbf{CT}' & \mathrm{r}\mathbf{I}_v \end{pmatrix}\begin{pmatrix} \mathbf{U} \\ \mathbf{T} \end{pmatrix}\right]. \tag{5.5}$$

The column (eliminating treatment and row effects) sum of squares is

$$\hat{\gamma}'\left[\mathbf{U}-(\mathbf{Z}\quad \mathbf{RT})\begin{pmatrix} \mathrm{k}\mathbf{I}_{rk} & \mathbf{RT}-\mathbf{J}_{rk\times v} \\ \mathbf{RT}' & \mathrm{r}\mathbf{I}_v \end{pmatrix}\begin{pmatrix} \mathbf{R} \\ \mathbf{T} \end{pmatrix}\right]. \tag{5.6}$$

The expected value for the mean square from (5.5) is

$$\sigma_\epsilon^2 + \mathrm{k}(\mathrm{r}-2)\sigma_\beta^2/(\mathrm{r}-1) = \tfrac{1}{3}(3\mathrm{err}+3\mathrm{rvc}) = \mathrm{err} + \mathrm{rvc} \tag{5.7}$$

from Table 5. The expected value of the sum of squares in (5.6) is

$$\mathrm{r}(\mathrm{k}-1)\Big(\sigma_\epsilon^2 + \mathrm{k}(\mathrm{r}-2)\sigma_\gamma^2/(\mathrm{r}-1) = 3(\mathrm{err}+\mathrm{col})\ , \tag{5.8}$$

as given in Table 5. The program is easily adjusted for resolvable r-row by c-column designs. The expected values above agree with those in Kempthorne (1952) and Federer (1955).

Table 5 displays the program for the example with v=4, k=2 and r=3. The symbols "(*" and "*)" enclose note in MATHEMATICA, which will not be executed. Other new commands are:

Join[r1, r2] -- concatenating r1 and r2, which can be scalars, vectors, matrices or arrays of higher dimension.

Module[{r1, r2}, expression] -- defining r1 and r2 as local variables that are valid only within the specifed expression, e.g., calculations, functions. Outside the expression, r1 and r2 do not have any specified values.

concatenate2[Z_, CT_] := -- the notation ":=" indicating a user-specified function to be defined, which was named as "concatenate2" here. The definition of the function is given after ":=", which can consist of operations, other build-in or user-specified functions. The square brackets enclose the arguments to the function. The names of the arguments are arbitrary, but the following underscore is necessary. Once the function is defined, any 2 arguments (underscores are not needed) with names other than Z and CT will be performed exactly the same way.

% -- this symbol representing the output or result from the previous command, not percentage.

6. LITERATURE CITED

Federer, W.T. (1955). *Experiment Design — Theory and Application.* MacMillan, New York (Republished as the Indian edition by Oxford & IBH Publishing Co., New Delhi, Calcutta, 1967).

Kempthorne, O. (1952). *The Design and Analysis of Experiments.* John Wiley & Sons, Inc., New York.

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(*TABLE 1. Mathematica program for evaluating the expected value
 of the error (a) sum of squares for a split plot experiment
 design with w whole plots in a RCBD of r replicates and
 s split plots.*)


In[32]:=
   r = 2; w = 3; s = 4;Array[Y,{r,w,s}];
   Y[g_,h_,i_] = m+x[g]+t[h]+d[g,h]+v[i]+tv[h,i]+e[g,h,i];
   res ={d[g_,h_] d[g_,h_]->D,d[g_,h_] d[i_,j_]->0,
   e[g_,h_,i_] e[g_,h_,i_]->E,e[g_,h_,i_] e[g_,j_,l_]->0,
   e[g_,h_,i_] d[g_,h_]->0,e[g_,h_,i_] d[j_,l_]->0,
   e[g_,h_,i_] e[j_,h_,i_]->0, e[g_,h_,i_] e[j_,l_,k_]->0 };
   (*Error A sum of squares is A.*)
   A = Sum[Sum[Y[g,h,i],{i,1,s}]^2/s,{g,1,r},{h,1,w}] -
   Sum[Sum[Y[g,h,i],{g,1,r},{i,1,s}]^2/(r s),{h,1,w}] -
   Sum[Sum[Y[g,h,i],{h,1,w},{i,1,s}]^2/(w s),{g,1,r}] +
   Sum[Y[g,h,i],{g,1,r},{h,1,w},{i,1,s}]^2/(r w s);
   Expand[A]/.res

Out[37]=
   8 D + 2 E
```

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(*Table 2.  Mathematica program for evaluating the expected
value of blocks eliminating treatment effects sum of squares
for an IBD with v = 4, k = 2, and r = 3, and the model is
Y[g,h,i] = m + x[g] + d[g,h] +t[i] + e[g,h,i].*)
```

In[63]:=

```
v = 4; r = 3; k = 2; S = Sum[t[i],{i,1,v}];
S1 = Sum[d[1,h],{h,1,k}]; S2 = Sum[d[2,h],{h,1,k}];
S3 = Sum[d[3,h],{h,1,k}];SBT = S1 + S2 + S3;
x11 = e[1,1,1] + e[1,1,2]; x12 = e[1,2,3] + e[1,2,4];
x21 = e[2,1,1] + e[2,1,3]; x22 = e[2,2,2] + e[2,2,4];
x31 = e[3,1,1] + e[3,1,4]; x32 = e[3,2,2] + e[3,2,3];
x1s = x11 + x12; x2s = x21 + x22; x3s = x31 + x32;
xs = x1s + x2s + x3s;
b11 = k d[1,1] + t[1] + t[2] + x11 - S1 - x1s/k;
b12 = k d[1,2] + t[3] + t[4] + x12 - S1 - x1s/k;
b21 = k d[2,1] + t[1] + t[3] + x21 - S2 - x2s/k;
b22 = k d[2,2] + t[2] + t[4] + x22 - S2 - x2s/k;
b31 = k d[3,1] + t[1] + t[4] + x31 - S3 - x3s/k;
b32 = k d[3,2] + t[2] + t[3] + x32 - S3 - x3s/k;
t1 = r t[1] + d[1,1] + d[2,1] + d[3,1] + e[1,1,1] + e[2,1,1] +
e[3,1,1] - xs/v - SBT/k;
t2 = r t[2] + d[1,1] + d[2,2] + d[3,2] + e[1,1,2] + e[2,2,2] +
e[3,2,2] - xs/v - SBT/k;
t3 = r t[3] + d[1,2] + d[2,1] + d[3,2] + e[1,2,3] + e[2,1,3] +
e[3,2,3] - xs/v - SBT/k;
t4 = r t[4] + d[1,2] + d[2,2] + d[3,1] + e[1,2,4] + e[2,2,4] +
e[3,1,4] - xs/v - SBT/k;
B = {{b11},{b12},{b21},{b22},{b31},{b32}};
T = {{t1},{t2},{t3},{t4}};
NB = {{1,1,0,0},{0,0,1,1},{1,0,1,0},{0,1,0,1},{1,0,0,1},
{0,1,1,0}};
Iv = IdentityMatrix[v]; Irk = IdentityMatrix[6];
Jv = Table[1,{i,1,v},{j,1,v}];
BN = Transpose[NB];
Ir = IdentityMatrix[r]; Jk = Table[1,{i,1,k},{j,1,k}];
J = {{1,1,0,0,0,0},{1,1,0,0,0,0},{0,0,1,1,0,0},
{0,0,1,1,0,0},{0,0,0,0,1,1},{0,0,0,0,1,1}};
Jrk = Table[1,{i,1,6},{j,1,6}] - J;
A = Inverse[k Irk - NB.BN/r + Jrk/r];
BeT = A.(B - NB.T/r);
res ={e[g_,h_,i_] e[g_,h_,i_]->E,e[g_,h_,i_] e[j_,k_,l_]->0,
d[g_,h_] d[g_,h_]->R,d[g_,h_] d[j_,k]->0,
e[g_,h_,i_] d[g_,h_]->0, e[g_,h_,i_] d[j_,k_]->0};
BssT = Expand[Transpose[BeT].(B - NB.T/r)]/.res
```

Out[93]=

```
{{3 E + 4 R}}
```

In[75]:=

```
(*TABLE 3. Mathematica program for evaluating the expected
value of the blocks eliminating treatment effects sum of
squares for an IBD with v = kb treatments in b incomplete
blocks of size k in n replicates.*)

n = 3; v = 6; k = 2; b = 3; r[g,h] = r[g,h];
SB1 = Sum[r[1,h],{h,1,b}];
SB2 = Sum[r[2,h],{h,1,b}];
SB3 = Sum[r[3,h],{h,1,b}]; S = SB1 + SB2 + SB3;
x11 = e[1,1,1] + e[1,1,4]; x12 = e[1,2,2] + e[1,2,5];
x13 = e[1,3,3] + e[1,3,6]; x1s = x11 + x12 + x13;
x21 = e[2,1,1] + e[2,1,5]; x22 = e[2,2,2] + e[2,2,6];
x23 = e[2,3,3] + e[2,3,4]; x2s = x21 + x22 + x23;
x31 = e[3,1,1] + e[3,1,6]; x32 = e[3,2,2] + e[3,2,4];
x33 = e[3,3,3] + e[3,3,5]; x3s = x31 + x32 + x33;
xs = x1s + x2s + x3s;
B11 = k r[1,1] - k SB1/b + x11 - x1s/b;
B12 = k r[1,2] - k SB1/b + x12 - x1s/b;
B13 = k r[1,3] - k SB1/b + x13 - x1s/b;
B21 = k r[2,1] - k SB2/b + x21 - x2s/b;
B22 = k r[2,2] - k SB2/b + x22 - x2s/b;
B23 = k r[2,3] - k SB2/b + x23 - x2s/b;
B31 = k r[3,1] - k SB3/b + x31 - x3s/b;
B32 = k r[3,2] - k SB3/b + x32 - x3s/b;
B33 = k r[3,3] - k SB3/b + x33 - x3s/b;
T1=r[1,1]+r[2,1]+r[3,1]-S/b+e[1,1,1]+e[2,1,1]+e[3,1,1]-xs/v;
T2=r[1,2]+r[2,2]+r[3,2]-S/b+e[1,2,2]+e[2,2,2]+e[3,2,2]-xs/v;
T3=r[1,3]+r[2,3]+r[3,3]-S/b+e[1,3,3]+e[2,3,3]+e[3,3,3]-xs/v;
T4=r[1,1]+r[2,3]+r[3,2]-S/b+e[1,1,4]+e[2,3,4]+e[3,2,4]-xs/v;
T5=r[1,2]+r[2,1]+r[3,3]-S/b+e[1,2,5]+e[2,1,5]+e[3,3,5]-xs/v;
T6=r[1,3]+r[2,2]+r[3,1]-S/b+e[1,3,6]+e[2,2,6]+e[3,1,6]-xs/v;
B = {{B11},{B12},{B13},{B21},{B22},{B23},{B31},{B32},{B33}};
T = {{T1},{T2},{T3},{T4},{T5},{T6}};
NT ={{1,0,0,1,0,0},
     {0,1,0,0,1,0},
     {0,0,1,0,0,1},
     {1,0,0,0,1,0},
     {0,1,0,0,0,1},
     {0,0,1,1,0,0},
     {1,0,0,0,0,1},
     {0,1,0,1,0,0},
     {0,0,1,0,1,0}};
J0 = {{0,0,0,1,1,1,1,1,1},
      {0,0,0,1,1,1,1,1,1},
      {0,0,0,1,1,1,1,1,1},
      {1,1,1,0,0,0,1,1,1},
      {1,1,1,0,0,0,1,1,1},
      {1,1,1,0,0,0,1,1,1},
```

```
        {1,1,1,1,1,1,0,0,0},
        {1,1,1,1,1,1,0,0,0},
        {1,1,1,1,1,1,0,0,0}};
  Irb = IdentityMatrix[9];
  TN = Transpose[NT];
  B0 = Inverse[k Irb - NT.TN/n + J0/n];
  res = {r[g_,h_] r[g_,h_]->R,r[g_,h_] r[i_,j_]->0,
  e[g_,h_,i_] e[g_,h_,i_]->E,e[g_,h_,i_] e[g_,h_,f_]->0,
  e[g_,h_,i_] e[g_,f_,i_]->0,e[g_,h_,i_] e[f_,h_,i_]->0,
  e[g_,h_,i_] e[c_,d_,f_]->0,e[g_,h_,i_] r[g_,h_]->0,
  e[g_,h_,i_] r[d_,h_]->0, e[g_,h_,i_] r[d_,f_]->0} ;
  Expand[Transpose[B - NT.T/n].B0.(B - NT.T/n)]/.res
```

Out[110]=

```
  {{6 E + 8 R}}
```

Table 4. *Mathematica program for evaluating the expectation of gradient eliminating treatment and block effects and block eliminating treatment and gradient effects sums of squares for an IBD with v = 9, k = 3, and r = 4.*

```
x11 = Sum[e[1,1,i],{i,1,3}];
x12 = Sum[e[1,2,i],{i,4,6}];
x13 = Sum[e[1,3,i],{i,7,9}];
x1s = x11 + x12 + x13;
x21 = e[2,1,1] + e[2,1,4] + e[2,1,7];
x22 = e[2,2,2] + e[2,2,5] + e[2,2,8];
x23 = e[2,3,3] + e[2,3,6] + e[2,3,9];
x2s = x21 + x22 + x23;
x31 = e[3,1,1] + e[3,1,5] + e[3,1,9];
x32 = e[3,2,2] + e[3,2,6] + e[3,2,7];
x33 = e[3,3,3] + e[3,3,4] + e[3,3,8];
x3s = x31 + x32 + x33;
xs = x1s + x2s + x3s;
S = Sum[t[i],{i,1,9}];
S1 = Sum[r[1,h],{h,1,3}];S2 =Sum[r[2,h],{h,1,3}];
S3 = Sum[r[3,h],{h,1,3}];S = S1 + S2 + S3;
b11 = 3 r[1,1]+t[1]+t[2]+t[3] - S1 + x11 - x1s/3;
b12 = 3 r[1,2]+t[4]+t[5]+t[6] - S1 + x12 - x1s/3;
b13 = 3 r[1,3]+t[7]+t[8]+t[9] - S1 + x13 - x1s/3;
b21 = 3 r[2,1]+t[1]+t[4]+t[7] - S2 + x21 - x2s/3;
b22 = 3 r[2,2]+t[2]+t[5]+t[8] - S2 + x22 - x2s/3;
b23 = 3 r[2,3]+t[3]+t[6]+t[9] - S2 + x23 - x2s/3;
b31 = 3 r[3,1]+t[1]+t[5]+t[9] - S3 + x31 - x3s/3;
b32 = 3 r[3,2]+t[2]+t[6]+t[7] - S3 + x32 - x3s/3;
b33 = 3 r[3,3]+t[3]+t[4]+t[8] - S3 + x33 - x3s/3;
t1 =  3 t[1]+r[1,1]+r[2,1]+r[3,1]+e[1,1,1]+e[2,1,1]+e[3,1,1]-
g[1,1]-g[2,1]-g[3,1]-Sum[r[g,h],{g,1,3},{h,1,3}]/3- xs/9;
t2 =  3 t[2]+r[1,1]+r[2,2]+r[3,2]+e[1,1,2]+e[2,2,2]+e[3,2,2]-
g[2,2] - Sum[r[g,h],{g,1,3},{h,1,3}]/3 - xs/9;
t3 =  3 t[3]+r[1,1]+r[2,3]+r[3,3]+e[1,1,3]+e[2,3,3]+e[3,3,3]+
g[1,1] - g[2,3] + g[3,3] -Sum[r[g,h],{g,1,3},{h,1,3}]/3-xs/9;
t4 =  3 t[4]+r[1,2]+r[2,1]+r[3,3]+e[1,2,4]+e[2,1,4]+e[3,3,4]-
g[1,2] - Sum[r[g,h],{g,1,3},{h,1,3}]/3 - xs/9;
t5 =  3 t[5]+r[1,2]+r[2,2]+r[3,1]+e[1,2,5]+e[2,2,5]+e[3,1,5]+
g[3,1] - Sum[r[g,h],{g,1,3},{h,1,3}]/3 - xs/9;
t6 =  3 t[6]+r[1,2]+r[2,3]+r[3,2]+e[1,2,6]+e[2,3,6]+e[3,2,6]+
g[1,2] - g[3,2] - Sum[r[g,h],{g,1,3},{h,1,3}]/3 - xs/9;
t7 =  3 t[7]+r[1,3]+r[2,1]+r[3,2]+e[1,3,7]+e[2,1,7]+e[3,2,7]-
g[1,3] + g[2,1] + g[3,2] -Sum[r[g,h],{g,1,3},{h,1,3}]/3-xs/9;
t8 =  3 t[8]+r[1,3]+r[2,2]+r[3,3]+e[1,3,8]+e[2,2,8]+e[3,3,8]+
g[2,2] - g[3,3] - Sum[r[g,h],{g,1,3},{h,1,3}]/3 - xs/9;
t9 =  3 t[9]+r[1,3]+r[2,3]+r[3,1]+e[1,3,9]+e[2,3,9]+e[3,1,9]+
g[1,3] + g[2,3] - Sum[r[g,h],{g,1,3},{h,1,3}]/3 - xs/9;
B = {{b11},{b12},{b13},{b21},{b22},{b23},{b31},{b32},{b33}};
```

```
T = {{t1},{t2},{t3},{t4},{t5},{t6},{t7},{t8},{t9}};
BT = {{b11},{b12},{b13},{b21},{b22},{b23},{b31},{b32},{b33},
{t1},{t2},{t3},{t4},{t5},{t6},{t7},{t8},{t9}};
g11 = 2 g[1,1] - t[1] + t[3] - e[1,1,1] + e[1,1,3];
g12 = 2 g[1,2] - t[4] + t[6] - e[1,2,4] + e[1,2,6];
g13 = 2 g[1,3] - t[7] + t[9] - e[1,3,7] + e[1,3,9];
g21 = 2 g[2,1] - t[1] + t[7] - e[2,1,1] + e[2,1,7];
g22 = 2 g[2,2] - t[2] + t[8] - e[2,2,2] + e[2,2,8];
g23 = 2 g[2,3] - t[3] + t[9] - e[2,3,3] + e[2,3,9];
g31 = 2 g[3,1] - t[1] + t[5] - e[3,1,1] + e[3,1,5];
g32 = 2 g[3,2] - t[6] + t[7] - e[3,2,6] + e[3,2,7];
g33 = 2 g[3,3] + t[3] - t[8] + e[3,3,3] - e[3,3,8];
G = {{g11},{g12},{g13},{g21},{g22},{g23},{g31},{g32},{g33}};
J0 =
{{0,0,0,1,1,1,1,1,1},{0,0,0,1,1,1,1,1,1},{0,0,0,1,1,1,1,1,1},
 {1,1,1,0,0,0,1,1,1},{1,1,1,0,0,0,1,1,1},{1,1,1,0,0,0,1,1,1},
 {1,1,1,1,1,1,0,0,0},{1,1,1,1,1,1,0,0,0},{1,1,1,1,1,1,0,0,0}};
NB = {{1,1,1,0,0,0,0,0,0},
      {0,0,0,1,1,1,0,0,0},
      {0,0,0,0,0,0,1,1,1},
      {1,0,0,1,0,0,1,0,0},
      {0,1,0,0,1,0,0,1,0},
      {0,0,1,0,0,1,0,0,1},
      {1,0,0,0,1,0,0,0,1},
      {0,1,0,0,0,1,1,0,0},
      {0,0,1,1,0,0,0,1,0}};
NG = {{-1, 0, 1,  0, 0, 0,  0, 0, 0},
      { 0, 0, 0, -1, 0, 1,  0, 0, 0},
      { 0, 0, 0,  0, 0, 0, -1, 0, 1},
      {-1, 0, 0,  0, 0, 0,  1, 0, 0},
      { 0,-1, 0,  0, 0, 0,  0, 1, 0},
      { 0, 0,-1,  0, 0, 0,  0, 0, 1},
      {-1, 0, 0,  0, 1, 0,  0, 0, 0},
      { 0, 0, 0,  0, 0,-1,  1, 0, 0},
      { 0, 0, 1,  0, 0, 0,  0,-1, 0}};
I9 = IdentityMatrix[9]; J1 = Table[1,{i,1,9},{j,1,9}];
BN = Transpose[NB];  GN = Transpose[NG];
A0B = Inverse[3 I9 - GN.NG/2 ];
A = Inverse[3 I9 - BN.NB/3 + J1/3];
B0B = Inverse[3 I9 - NB.A0B.BN + J0/3];
B0G = Inverse[2 I9 - NG.A.GN];
RHG = G - NG.A.(T - BN.B/3);
RHB = B - NB.A0B.(T - GN.G/2);
Expand[T - GN.G/2];
Expand[T - BN.B/3];
GTe = B0G.RHG;
BTe = B0B.RHB;
rgs = {g[f_,h_] g[f_,h_]->GR, g[f_,h_] g[i_,j_]->0,
e[g_,h_,i_] e[g_,h_,i_]->E, e[g_,h_,i_] e[g_,h_,f_]->0,
```

```
e[g_,f_,d_] e[g_,h_,i_]->0,e[g_,h_,i_] e[f_,h_,i_]->0,
e[g_,h_,i_] e[d_,h_,i_]->0,e[g_,h_,i_] e[c_,d_,f_]->0,
e[g_,h_,i_] g[g_,h_]->0,e[g_,h_,i_] g[f_,d_]->0};
rbs = {r[g_,h_] r[g_,h_]->BL, r[g_,h_] r[i_,j_]->0,
e[g_,h_,i_] e[g_,h_,i_]->E, e[g_,h_,i_] e[g_,h_,f_]->0,
e[g_,f_,d_] e[g_,h_,i_]->0,e[g_,h_,i_] e[f_,h_,i_]->0,
e[g_,h_,i_] e[d_,h_,i_]->0,e[g_,h_,i_] e[c_,d_,f_]->0,
e[g_,h_,i_] r[g_,h_]->0,e[g_,h_,i_] r[f_,d_]->0};
Expand[Transpose[GTe].RHG]/.rgs
Expand[Transpose[BTe].RHB]/.rbs
```

Out[319]=

```
{{9 E + 10 GR}}
```

Out[320]=

$$\{\{\frac{2707\ BL}{390} + 6\ E\}\}$$

Table 5. Mathematica program for obtaining the expected values of the row eliminating treatment and column effects and of the column eliminating row and treatment effects for a lattice square design with v = 4, k = 2, and r = 3. The linear model used is
Y[g,h,i,j] = m + b[g] + d[g,h] + c[g,i] + t[j] + e[g,h,i,j].)*

In[74]:=

```
v = 4; k = 2; r = 3;
xr11 = e[1,1,1,1] + e[1,1,2,2];xr12 = e[1,2,1,3] + e[1,2,2,4];
xr21 = e[2,1,1,1] + e[2,1,2,3];xr22 = e[2,2,1,4] + e[2,2,2,2];
xr31 = e[3,1,1,1] + e[3,1,2,4];xr32 = e[3,2,1,2] + e[3,2,2,3];
wc11 = e[1,1,1,1] + e[1,2,1,3];wc12 = e[1,1,2,2] + e[1,2,2,4];
wc21 = e[2,1,1,1] + e[2,2,1,4];wc22 = e[2,1,2,3] + e[2,2,2,2];
wc31 = e[3,1,1,1] + e[3,2,1,2];wc32 = e[3,1,2,4] + e[3,2,2,3];
xrs1 = xr11 + xr12; xrs2 = xr21 + xr22; xrs3 = xr31 + xr32;
wcs1 = wc11 + wc12; wcs2 = wc21 + wc22; wcs3 = wc31 + wc32;
xrs = xrs1 + xrs2 + xrs3; wcs = wcs1 + wcs2 + wcs3;
St = Sum[t[j],{j,1,v}]; Sr1 = Sum[d[1,h],{h,1,k}];
Sr2 = Sum[d[2,h],{h,1,k}]; Sr3 = Sum[d[3,h],{h,1,k}];
Sc1 = Sum[c[1,i],{i,1,k}]; Sc2 = Sum[c[2,i],{i,1,k}];
Sc3 = Sum[c[3,i],{i,1,k}];
Srs = Sr1 + Sr2 + Sr3; Scs = Sc1 + Sc2 + Sc3;
a11 = k d[1,1] + t[1] + t[2] + xr11 - Sr1 - xrs1/k - St/k;
a12 = k d[1,2] + t[3] + t[4] + xr12 - Sr1 - xrs1/k - St/k;
a21 = k d[2,1] + t[1] + t[3] + xr21 - Sr2 - xrs2/k - St/k;
a22 = k d[2,2] + t[4] + t[2] + xr22 - Sr2 - xrs2/k - St/k;
a31 = k d[3,1] + t[1] + t[4] + xr31 - Sr3 - xrs3/k - St/k;
a32 = k d[3,2] + t[2] + t[3] + xr32 - Sr3 - xrs3/k - St/k;
b11 = k c[1,1] + t[1] + t[3] + wc11 - Sc1 - wcs1/k - St/k;
b12 = k c[1,2] + t[2] + t[4] + wc12 - Sc1 - wcs1/k - St/k;
b21 = k c[2,1] + t[1] + t[4] + wc21 - Sc2 - wcs2/k - St/k;
b22 = k c[2,2] + t[3] + t[2] + wc22 - Sc2 - wcs2/k - St/k;
b31 = k c[3,1] + t[1] + t[2] + wc31 - Sc3 - wcs3/k - St/k;
b32 = k c[3,2] + t[4] + t[3] + wc32 - Sc3 - wcs3/k - St/k;
t1 = r t[1]+d[1,1]+c[1,1]+d[2,1]+c[2,1]+d[3,1]+c[3,1]-r St/v +
e[1,1,1,1]+e[2,1,1,1]+e[3,1,1,1] - xrs/v - Srs/k - Scs/k;
t2 = r t[2]+d[1,1]+c[1,2]+d[2,2]+c[2,2]+d[3,2]+c[3,1]-r St/v +
e[1,1,2,2]+e[2,2,2,2]+e[3,2,1,2] - xrs/v - Srs/k - Scs/k;
t3 = r t[3]+d[1,2]+c[1,1]+d[2,1]+c[2,2]+d[3,2]+c[3,2]-r St/v +
e[1,2,1,3]+e[2,1,2,3]+e[3,2,2,3] - xrs/v - Srs/k - Scs/k;
t4 = r t[4]+d[1,2]+c[1,2]+d[2,2]+c[2,1]+d[3,1]+c[3,2]-r St/v +
e[1,2,2,4]+e[2,2,1,4]+e[3,1,2,4] - xrs/v - Srs/k - Scs/k;
R = {{a11},{a12},{a21},{a22},{a31},{a32}};
U = {{b11},{b12},{b21},{b22},{b31},{b32}};
T = {{t1},{t2},{t3},{t4}};
Ir = IdentityMatrix[r];
```

```
Iv = IdentityMatrix[v];
Irk = IdentityMatrix[r k];
Jk = Table[1,{i,1,2},{j,1,2}];
Jrk = Table[1,{i,1,r k},{j,1,r k}];
Jv = Table[1,{i,1,v},{j,1,v}];
Jcv = Table[1,{i,1,r k},{j,1,v}];
(* RC = Outer[Times,Ir,Jk] is the Kronecker product of two
matrices but it doesn't seem to work here so it will be
written as RC minus the restrictions or the following Z:*)
Z = Table[0,{i,1,6},{j,1,6}];
RT = {{1,1,0,0},{0,0,1,1},{1,0,1,0},{0,1,0,1},{1,0,0,1},
{0,1,1,0}};
CT = {{1,0,1,0},{0,1,0,1},{1,0,0,1},{0,1,1,0},{1,1,0,0},
{0,0,1,1}};
TR = Transpose[RT]; TC = Transpose[CT];
m = k Irk; n = r Iv;
o = CT - Jcv; s = RT - Jcv;
concatanate[Z_,CT_]:=
      Module[{r1},
      r1 = Transpose[ Join[Transpose[Z], Transpose[CT]]];
      Join[r1]
]
concatanate[Z,CT];
MC1 = %;
concatanate[Z_,RT_]:=
      Module[{r1},
      r1 = Transpose[ Join[Transpose[Z], Transpose[RT]]];
      Join[r1]
]
concatanate[Z,RT];
MR2 = %;
MC3 = {{b11},{b12},{b21},{b22},{b31},{b32},
{t1},{t2},{t3},{t4}};
MR4 = Transpose[{a11,a12,a21,a22,a31,a32,t1,t2,t3,t4}];
concatanate[m_,o_,TC_,n_]:=
      Module[{r1,r2},
      r1 = Transpose[ Join[Transpose[m], Transpose[o]]];
      r2 = Transpose[ Join[Transpose[TC], Transpose[n]]];
      Join[r1,r2]
]
concatanate[m,o,TC,n];
MC5 = %;
concatanate[m_,s_,TR_,n_]:=
      Module[{r1,r2},
      r1 = Transpose[ Join[Transpose[m], Transpose[s]]];
      r2 = Transpose[ Join[Transpose[TR], Transpose[n]]];
      Join[r1,r2]
]
concatanate[m,s,TR,n];
```

```
MR6 = %;
MC7 = Inverse[k Irk - MC1.Inverse[MR6].Transpose[MC1] +
Jrk/r];
MR8 = Inverse[k Irk - MR2.Inverse[MC5].Transpose[MR2] +
Jrk/r];
MC9 = U - MC1.Inverse[MR6].MR4;
MR0 = R - MR2.Inverse[MC5].MC3;
res1 = {c[g_,i_] c[g_,i_]->col, c[g_,i_] c[f_,l_]->0,
e[g_,h_,i_,j_] e[g_,h_,i_,j_]->err,
e[g_,h_,i_,j_] c[g_,i_]->0, e[g_,h_,i_,j_] c[f_,l_]->0,
e[g_,h_,i_,j_] e[w_,x_,y_,z_]->0};
Expand[Transpose[MC7.MC9].MC9];
%/.res1
res2 = {d[g_,h_] d[g_,h_]->rvc, d[g_,h_] d[f_,l_]->0,
e[g_,h_,i_,j_] e[g_,h_,i_,j_]->err,
e[g_,h_,i_,j_] d[g_,h_,]->0,e[g_h_,i_,j_] d[f_,l_]->0,
e[g_,h_,i_,j_] e[w_,x_,y_,z_]->0,c[g_,i_] d[g_,h_]->0,
c[g_,i_] d[f_,h_]->0,d[g_,h_] e[g_,h_,i_,j_]->0,
d[g_,h_] e[f_,l_, i_,j_]->0};
Expand[Transpose[MR8.MR0].MR0];
%/.res2
```

Out[142]=

```
{{3 col + 3 err}}
```

Out[145]=

```
{{3 err + 3 rvc}}
```